Math 2010 Week 4

Vector-valued functions of multivariables

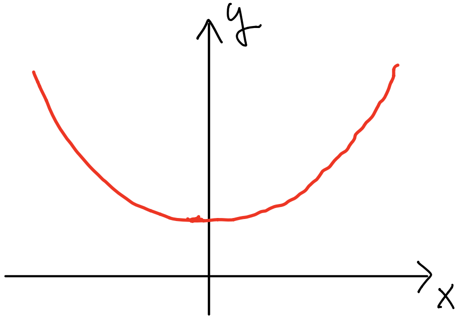
$\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$. How to visualize it?

① Graph of $\vec{f}$

$$\text{Graph}(f) = \{ (\underbrace{\vec{x}}_{\text{in } \mathbb{R}^n}, \underbrace{\vec{f}(\vec{x})}_{\text{in } \mathbb{R}^m}) \in \mathbb{R}^{n+m} : \vec{x} \in \Omega \} \subseteq \mathbb{R}^{n+m}$$

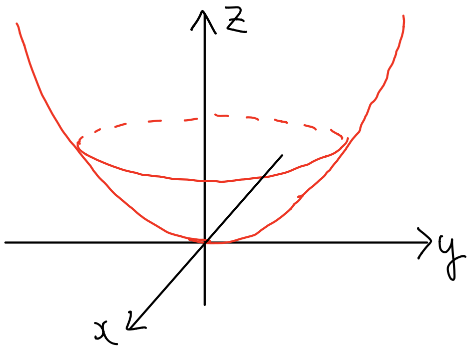
eg

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = 1 + x^2$



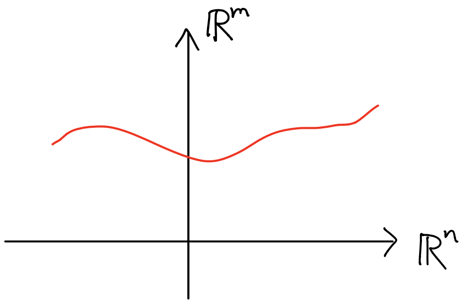
$\text{Graph}(f) \subseteq \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x^2 + y^2$



$\text{Graph}(g) \subseteq \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$

In general $\vec{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$



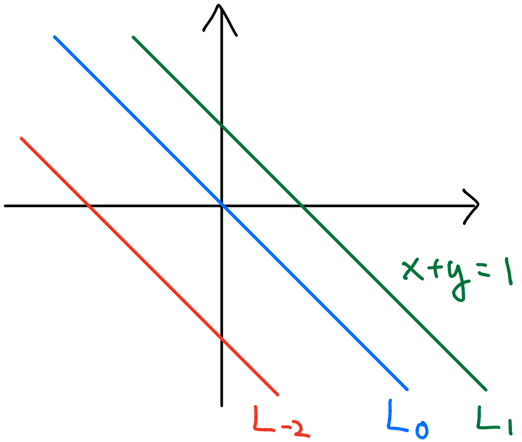
Hard to draw if $n+m > 3$

② Level set of $\vec{f}: \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

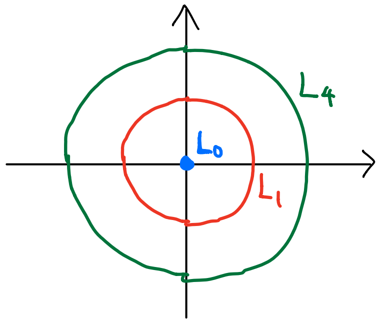
If $c \in \mathbb{R}^m$, define the level set at c to be

$$L_c = \{ x \in \Omega : \vec{f}(x) = \vec{c} \} = \vec{f}^{-1}(\vec{c}) \subseteq \Omega \subseteq \mathbb{R}^n$$

eg $f(x,y) = x+y \quad \Omega = \mathbb{R}^2$

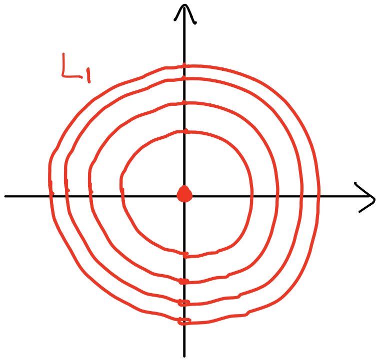


eg $g(x,y) = x^2+y^2 \quad \Omega = \mathbb{R}^2$



L_c is $\begin{cases} \phi & \text{if } c<0 \\ \text{a point} & \text{if } c=0 \\ \text{a circle} & \text{if } c>0 \end{cases}$

eg $h(x,y) = \cos\left(2\pi(x^2+y^2)\right) \quad \Omega = \mathbb{R}^2$



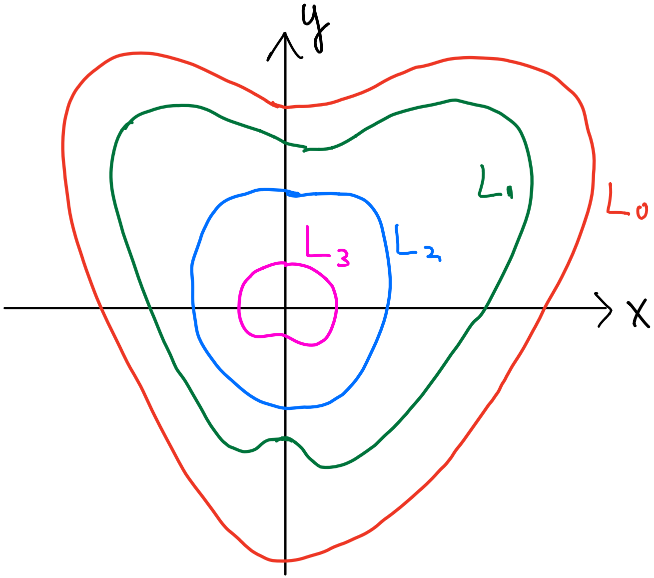
$h(x,y) = 1 \iff x^2+y^2 \in \mathbb{Z}$

$\therefore L_1$ consists of infinitely many circles with radius $\sqrt{k}$, $k \geq 1$ and the origin

$$L_1 = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \in \mathbb{Z} \}$$

Relate level set and graph

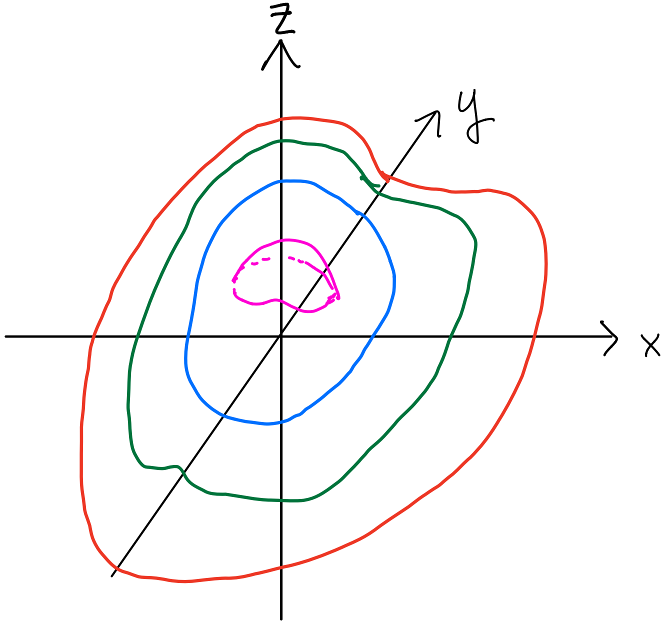
eg $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$



Level sets of f

(drawn on domain $= \mathbb{R}^2$)

$\sim$ contour lines a map



Graph of f

(drawn on domain $\times$ codomain $= \mathbb{R}^3$)

$\sim$ Mountain

Limit of multi-variable functions

Let $A \subseteq \mathbb{R}^n$. Define (Au 2.5, 2.6 Thomas 14.2)

$\overline{A} = A \cup \partial A$ = closure of A

For $a \in \overline{A}$ and $\vec{f}: A \to \mathbb{R}^m$

Want to define $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{L}$ to mean

If $\vec{x}$ is very close to $\vec{a}$,

$\vec{f}(\vec{x})$ is very close to $\vec{L}$

Close $\longleftrightarrow$ distance is small

Defn (ε-δ) $\vec{f}: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

We say that $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ if

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $\vec{x} \in A$ and $0 < \|\vec{x} - \vec{a}\| < \delta$

then $\|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon$

Rmk ① $\forall$ = for all $\quad \exists$ = there exists

② $\|\vec{x} - \vec{a}\|$ = distance between $\vec{x}$ and $\vec{a}$ in $\mathbb{R}^n$

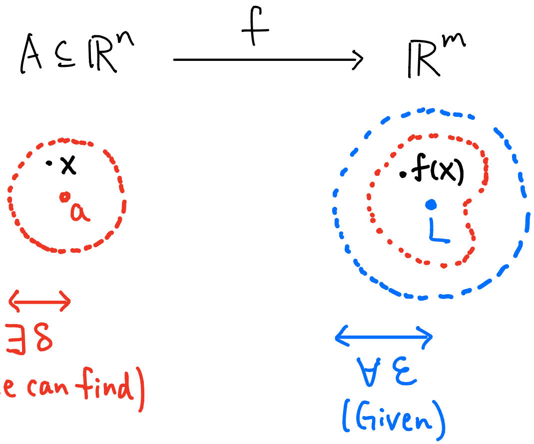
$0 < \|\vec{x} - \vec{a}\| < \delta$
$\uparrow$
means $\vec{x} \neq \vec{a}$

$\therefore$ Consider points close to $\vec{a}$ but not equal to $\vec{a}$

③ $\|\vec{f}(\vec{x}) - \vec{L}\|$ = distance between $f(\vec{x})$ and $\vec{L}$ in $\mathbb{R}^m$

If $m = 1$, $\|\vec{f}(\vec{x}) - \vec{L}\| = |f(\vec{x}) - L| \leftarrow$ absolute value

Picture



eg $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x+y$

Illustrate that $\displaystyle\lim_{(x,y)\to(1,2)} f(x,y) = 3$

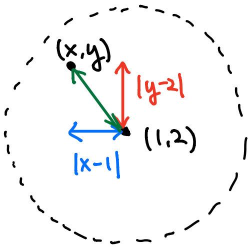
i.e.

$\boxed{\begin{array}{l}\text{Show that given any } \varepsilon>0, \\ \text{one can find } \delta>0 \text{ such that} \\ \text{if } 0<\|(x,y)-(1,2)\|<\delta, \\ \text{then } |f(x,y)-3|<\varepsilon.\end{array}}$

Idea: $|f(x,y)-3| = |(x-1)+(y-2)|$

$\leq |x-1| + |y-2|$

$\|(x,y)-(1,2)\| = \sqrt{(x-1)^2+(y-2)^2}$



For example, for $\varepsilon = 1$, one can pick $\delta = \frac{1}{2}$:

If $\|(x,y)-(1,2)\| < \delta = \frac{1}{2}$, then

$$|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2+(y-2)^2} < \frac{1}{2}$$

$$|y-2| = \sqrt{(y-2)^2} \leq \sqrt{(x-1)^2+(y-2)^2} < \frac{1}{2}$$

$$\Rightarrow |f(x,y)-3| \leq |x-1| + |y-2| < \frac{1}{2} + \frac{1}{2} = 1 = \varepsilon \ \checkmark$$

Similarly, for $\varepsilon = \frac{1}{100}$, one can pick $\delta = \frac{1}{200}$

In general, we need to do it for any $\varepsilon > 0$

For any given $\varepsilon > 0$, one can pick $\delta = \frac{\varepsilon}{2}$. Then

$$\|(x,y)-(1,2)\| < \delta = \frac{\varepsilon}{2}$$

$$\Rightarrow |f(x,y)-3| = |x+y-3| \leq |x-1| + |y-2|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\therefore \lim_{(x,y)\to(1,2)} f(x,y) = 3$$

eg Let $f(x,y) = x^2+y^2$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ from definition

Sol Need to show that $\forall \varepsilon > 0$,

$\exists \delta > 0$ such that

if $0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$

then $\|f(x,y) - 0\| = |x^2+y^2| < \varepsilon$

eg If $\varepsilon = \frac{1}{100}$,

one can pick $\delta = \frac{1}{10}$ (or anything smaller)

In general, given $\varepsilon > 0$, one can pick $\delta = \sqrt{\varepsilon}$

Then for $0 < \|(x,y)-(0,0)\| < \delta$

$$\|f(x,y)-(0,0)\| = |x^2+y^2| = \left(\sqrt{x^2+y^2}\right)^2 < \delta^2 = \varepsilon$$

Let $A \subseteq \mathbb{R}^n$, $\vec{a} \in \bar{A}$, $\vec{f}: A \to \mathbb{R}^m$

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$

Each $f_i : A \to \mathbb{R}$ is called a component of $\vec{f}$

Prop

$$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix} \iff \lim_{\vec{x}\to\vec{a}} f_i(\vec{x}) = l_i \text{ for } i = 1, 2, \cdots, m$$

Consequence:

It is good enough for us to focus on limit of real-valued functions $f: A \to \mathbb{R}$ $(m=1)$

eg $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$

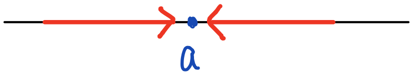
$$\vec{f}(x,y) = \begin{bmatrix} x+y \\ x^2+y^2+1 \end{bmatrix} \begin{array}{l} \leftarrow f_1(x,y) \\ \leftarrow f_2(x,y) \end{array}$$

$$\lim_{(x,y)\to(1,2)} \vec{f}(x,y) = \begin{bmatrix} \lim_{(x,y)\to(1,2)} x+y \\ \lim_{(x,y)\to(1,2)} x^2+y^2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Limit along a path

In one variable:

Two ways to approach $a \in \mathbb{R}$

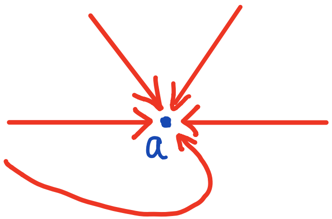


$\lim_{x \to a} f(x)$ exists $\Leftrightarrow$ $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

(exist and equal)

For n variables, $n \geq 2$

Many ways to approach $a \in \mathbb{R}^n$



The situation is not as simple

Need to consider all different curves to a

Fact $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $\vec{a} \in \bar{A}$

$$\boxed{\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L \iff \text{limit of } f(\vec{x}) \text{ when } \vec{x} \text{ approaches to } \vec{a} \text{ along any path exists and equals to } L}$$

Useful for showing limit does not exists (DNE)

- Find one path such that the limit along that path DNE

or

- Find two paths such that the limits along the two paths are different

$\Rightarrow \lim_{\vec{x} \to \vec{a}} f(\vec{x})$ DNE eg $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ ← the function is not defined at $(0,0)$

Sol Look at limits along different paths

① Along x-axis $(y=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2}$$

$$= \lim_{x\to 0} 1 = 1$$

② Along y-axis $(x=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{0^2-y^2}{0^2+y^2}$$

$$= \lim_{y\to 0} -1 = -1 \neq 1$$

$\therefore$ Different limits along different paths

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ DNE}$$

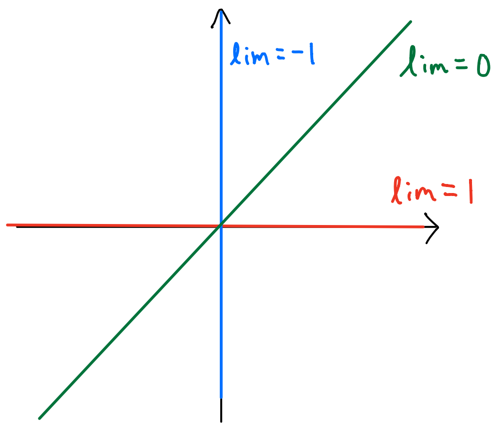
Rmk We may look at other paths too

eg. Along $y=x$ (45°)

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-x^2}{x^2+x^2}$$

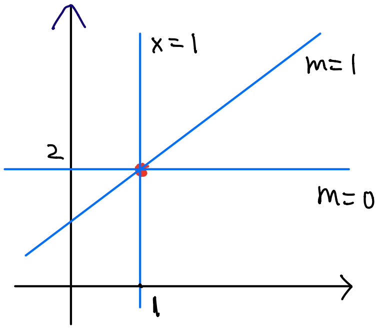
$$= \lim_{x\to 0} 0 = 0$$

What is limit along different slopes?



eg $\displaystyle \lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$

Sol Find limit along different lines



slope $= m$

① Along $x=1$

$$\lim_{\substack{(x,y)\to(1,2)\\ x=1}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$$

$$= \lim_{y\to 2} \frac{(1)y-2(1)-y+2}{(1-1)^2+(y-2)^2}$$

$$= \lim_{y\to 2} 0 = 0$$

② Along $y-2 = m(x-1)$

$$\lim_{\substack{(x,y)\to(1,2)\\ y-2=m(x-1)}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$$

$$= \lim_{\substack{(x,y)\to(1,2)\\ y-2=m(x-1)}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}$$

$$= \lim_{x\to 1} \frac{m(x-1)^2}{(x-1)^2+m^2(x-1)^2}$$

$$= \frac{m}{1+m^2}$$ ← different limits for different m

eg. If $m=1$, limit $= \frac{1}{2}$

If $m=0$, limit $= 0$

$$\therefore \lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2} \text{ DNE}$$

eg $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

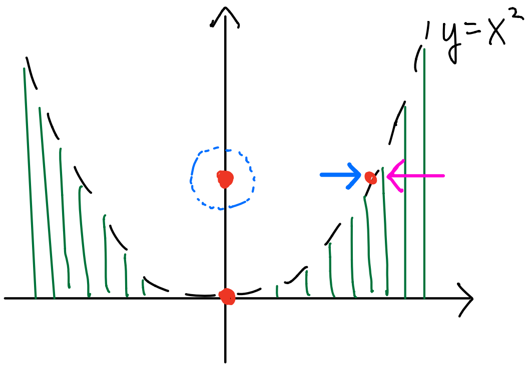
$$f(x,y) = \begin{cases} 1 & \text{if } 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

Find $\lim\limits_{(x,y)\to\vec{a}} f(x,y)$, where

i. $\vec{a} = (0,1)$

ii. $\vec{a} = (1,1)$

iii $\vec{a} = (0,0)$



$f \equiv 1$ on ▥ $\quad f \equiv 0$ otherwise

Sol i $f \equiv 0$ near $(0,1) \Rightarrow \lim\limits_{(x,y)\to(0,1)} f(x,y) = 0$

ii $\lim\limits_{\substack{(x,y)\to(1,1) \\ y=1,\, x<1}} f(x,y) = 0 \qquad \lim\limits_{\substack{(x,y)\to(1,1) \\ y=1,\, x>1}} f(x,y) = 1$

different

$\Rightarrow \lim\limits_{(x,y)\to(1,1)} f(x,y)$ DNE

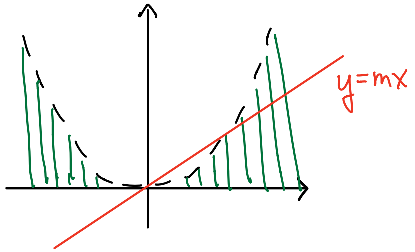
iii **Case 1** Along y-axis $(x=0)$

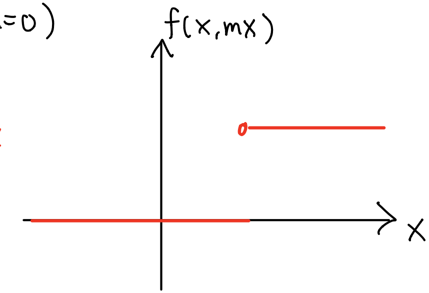
$f \equiv 0$ on y-axis $\Rightarrow \lim\limits_{\substack{(x,y)\to(0,0) \\ x=0}} f(x,y) = 0$

Case 2 Along $y = mx$

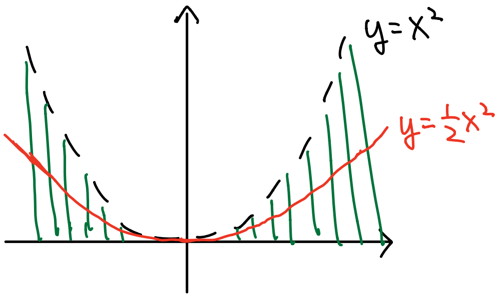
$$\lim_{\substack{(x,y)\to(0,0) \\ y=mx}} f(x,y) = \lim_{x\to 0} f(x,mx) = 0$$

If $m > 0$ (similar for $m<0$, $m=0$)





Case 3 Along the curve $y=\frac{1}{2}x^2$



$$\lim_{\substack{(x,y)\to(0,0)\\ y=\frac{1}{2}x^2}} f(x,y) = \lim_{x\to 0} f\left(x,\tfrac{1}{2}x^2\right) \overset{*}{=} 1$$

$\neq 0$ as in case 1, 2

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ DNE

$$* \quad f\left(x,\tfrac{1}{2}x^2\right) = \begin{cases} 1 & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

Rmk Another way to show limit DNE is by ε-δ argument (Ex)

Properties of Limits

Assuming all limits on the right hand side exist then the limit on the left hand side exists and the formula holds

1. $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm \lim_{\vec{x}\to\vec{a}} g(\vec{x})$

2. $\lim_{\vec{x}\to\vec{a}} k f(\vec{x}) = k \lim_{\vec{x}\to\vec{a}} f(\vec{x})$ where k is a constant.

3. $\lim_{\vec{x}\to\vec{a}} f(\vec{x})\, g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \cdot \lim_{\vec{x}\to\vec{a}} g(\vec{x})$

4. $\lim_{\vec{x}\to\vec{a}} \dfrac{f(\vec{x})}{g(\vec{x})} = \dfrac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}\to\vec{a}} g(\vec{x})}$ if $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$

5. $\lim_{\vec{x}\to\vec{a}} [f(\vec{x})]^n = \left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)^n$, $n \geq 0$

6. $\lim_{\vec{x}\to\vec{a}} [f(\vec{x})]^{\frac{1}{n}} = \left[\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right]^{\frac{1}{n}}$ (If n is even, assume $f(\vec{x}) \geq 0$ near $\vec{a}$)

Squeeze theorem (Sandwich theorem)

Let $f, g, h : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

If $g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x})$ near $a \in \Omega$

and $\lim\limits_{\vec{x}\to\vec{a}} g(\vec{x}) = \lim\limits_{\vec{x}\to\vec{a}} h(\vec{x}) = L$

Then $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = L$

Rmk We say that a statement

$P(\vec{x})$ is true near $\vec{a} \in \mathbb{R}^n$ if

$P(\vec{x})$ is true $\forall \vec{x} \in D_\delta(\vec{a}) \setminus \{\vec{a}\}$ for some $\delta > 0$

Note $|f(\vec{x})| \leq g(\vec{x}) \Rightarrow -g(\vec{x}) \leq f(\vec{x}) \leq g(\vec{x})$

Hence,

Special case of Squeeze theorem

$|f(\vec{x})| \leq g(\vec{x})$ near $\vec{a}$ and $\lim\limits_{\vec{x}\to\vec{a}} g(\vec{x}) = 0$

$\Rightarrow \lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = 0$

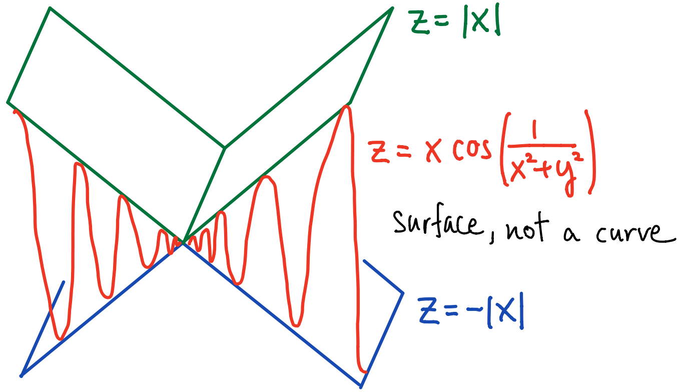
eg $\lim\limits_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right)$

Sol Note

$$\left|\cos\left(\frac{1}{x^2+y^2}\right)\right| \leq 1 \Rightarrow \left|x\cos\left(\frac{1}{x^2+y^2}\right)\right| \leq |x|$$

Also, $\lim\limits_{(x,y)\to(0,0)} |x| = 0$

$\therefore \lim\limits_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right) = 0$ by squeeze thm



eg2 $$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}$$

Sol Note

$$\left|\frac{(x-1)^2 \ln x}{(x-1)^2+y^2}\right| = \left|\frac{(x-1)^2}{(x-1)^2+y^2}\right| \cdot |\ln x|$$

$$\le |\ln x|$$

Also, $\displaystyle\lim_{(x,y)\to(1,0)} |\ln x| = |\ln(1)| = 0$

By squeeze theorem,

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0$$

Rmk If $a \le b$, then

$ca \le cb$ if $c > 0$

$ca \le cb$ if $c < 0$